Time decay of the solutions of the Vlasov-Poisson system in the plasma physical case

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Abstract
We introduce a new identity satisfied by solutions of the Vlasov-Poisson system. It has the property that all quantities which appear have a definite sign, and this allows us to prove new results on the time decay of the solutions in the plasma physical case.

1 Introduction and main results
The Vlasov-Poisson system
\[ \partial_t f + v \cdot \partial_x f - \partial_x U \cdot \partial_v f = 0, \]
\[ \Delta U = 4\pi \gamma \rho, \quad \rho(t, x) = \int f(t, x, v) dv \]
describes for \( \gamma = -1 \) the time evolution of a collisionless plasma of electrons or ions which interact via the Coulomb field generated by the plasma itself. Here \( f = f(t, x, v) \) denotes the number density of the electrons or ions on phase space, \( t \geq 0, \, x, v \in \mathbb{R}^3 \) denote time, position, and velocity respectively, \( \rho = \rho(t, x) \) denotes the spatial charge density of the plasma, and \( U = U(t, x) \) denotes the Coulomb potential. In the so-called stellar dynamical case \( \gamma = 1 \) the system describes a selfgravitating, collisionless ensemble such as a galaxy.
The question whether classical solutions to the corresponding initial value problem exist globally in time has long been open and was answered positively for general initial data by Pfaffelmoser [8]. Simplified versions of the proof with stronger estimates on the possible growth of the solutions are due to Schaeffer [9] and Horst [5], a proof along different lines was given by Lions and Perthame [7]. All these proofs work for both the plasma physical and the stellar dynamical cases and in both cases produce the same growth estimates. However, from a physics point of view one expects a plasma to spread out and consequently its spatial density to decay as opposed to the stellar dynamical case, where more interesting behaviour is to be expected. This is substantiated in [2] and [4], where some decay results are shown in the plasma physical case.

In the present note we prove the following results:

**Theorem 1** Let \( f \) be a classical solution of the Vlasov-Poisson system with nonnegative initial datum \( f(0) \in C^1_c(\mathbb{R}^6) \). Then the following identity holds:

\[
\frac{d}{dt} \int \int (x-tv)^2 f(t,x,v) dv dx = \frac{\gamma}{4\pi} \left( t^2 \frac{d}{dt} \| \partial_x U(t) \|_2^2 + t \| \partial_x U(t) \|_2^2 \right), \quad t \geq 0.
\]

(1)

**Theorem 2** In the plasma physical case \( \gamma = -1 \) there exists for every solution of the Vlasov-Poisson system with initial datum as above a constant \( C > 0 \) such that the following estimates hold:

\[
\| \partial_x U(t) \|_2 \leq C(1+t)^{-1/2}, \quad t \geq 0.
\]

(2)

\[
\int \int (x-tv)^2 f(t,x,v) dv dx \leq C(1+t), \quad t \geq 0.
\]

(3)

\[
\| \rho(t) \|_{5/3} \leq C(1+t)^{-3/5}, \quad t \geq 0.
\]

(4)

We prove these results in the next section. Both theorems remain valid if the plasma contains different particle species with possibly different signs of charge. Note that no decay can be expected of general solutions in the stellar dynamical case since stationary solutions are known to exist [1] in that case. The estimate (2) has already been obtained in [2] using the so-called dilation identity

\[
\frac{d}{dt} \int \int x \cdot v f(t,x,v) dv dx = \int \int v^2 f(t,x,v) dv dx - \frac{\gamma}{8\pi} \| \partial_x U(t) \|_2^2.
\]

(5)

Our identity can be derived from the dilation identity but we give a direct proof below, which is quite short anyway. The positive definiteness of the left hand side in (3) allows us to obtain also the decay of \( \| \rho(t) \|_{5/3} \). So far only the boundedness of the latter quantity was known, a result due to Horst [3] for \( \gamma = \pm 1 \). Since the boundedness of \( \| \rho(t) \|_{5/3} \) plays a major role in the global
existence results we conjecture that the time decay of this quantity should lead to improvements in the growth estimates for $\gamma = -1$.

Our search for an identity like (1) was motivated by [6], where its “quantum mechanical counterpart” is used to obtain global existence and time decay for solutions of the Wigner-Poisson system, a system which takes into account quantum corrections to the motion of the plasma particles.

2 Proofs of Theorems 1 and 2

As proved in Refs. [3], [8], or [9], our assumption that $f(0)$ be compactly supported implies that the same is true for $f(t)$ for all $t \geq 0$. Thus all integrals which we consider below exist, and integrations by parts produce no boundary terms. This could also be ensured by appropriate fall-off conditions in $x$ and $v$.

Proof of Theorem 1: Using the Vlasov equation and integrating by parts we obtain

$$\frac{d}{dt} \int (x - tv)^2 f(t,x,v) dv \, dx$$

$$= -2 \int (x - tv) \cdot vf \, dv \, dx - \int (x - tv)^2 (v \cdot \partial_x f - \partial_x U \cdot \partial_v f) \, dv \, dx$$

$$= 2t \int x \cdot \partial_x U \, \rho \, dx - 2t^2 \int \partial_x U \cdot j \, dx$$

where

$$j(t,x) = \int v f(t,x,v) \, dv.$$

Now

$$\int x \cdot \partial_x U \, \rho \, dx = \gamma \int \frac{x - y}{|x - y|^3} \rho(t,y) \rho(t,x) \, dy \, dx$$

$$= \gamma \int \frac{\rho(t,y)}{|x - y|^3} \rho(t,x) \, dx + \gamma \int \frac{y - x}{|x - y|^3} \rho(t,y) \rho(t,x) \, dx$$

$$= - \int U \, \rho \, dx - \int y \cdot \partial_y U \, \rho \, dy,$$

and thus

$$2 \int x \cdot \partial_x U \, \rho \, dx = - \int U \, \rho \, dx = \frac{\gamma}{4\pi} \int |\partial_x U|^2 \, dx.$$

Integration by parts and the continuity equation $\partial_t \rho + \text{div} j = 0$ (which results from integration of the Vlasov equation with respect to $v$) yield

$$\int \partial_x U \cdot j \, dx = - \int U \, \text{div} j \, dx = \int U \, \partial_t \rho \, dx = \frac{1}{2} \frac{d}{dt} \int U \, \rho \, dx = - \frac{\gamma}{8\pi} \frac{d}{dt} \int |\partial_x U|^2 \, dx;$$

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note that

\[ \int \partial_t U \rho \, dx = \gamma \iint \frac{\partial_t \rho(t, y) \rho(t, x)}{|x - y|} \, dy \, dx = \int U \partial_t \rho \, dy. \]

We collect terms, and the proof of Theorem 1 is complete.

**Proof of Theorem 2:** We define

\[ g(t) = \frac{t^2}{4\pi} \| \partial_x U(t) \|_2^2 \]

and rewrite the identity (1) for \( \gamma = -1 \) in the form

\[ \frac{d}{dt} \left[ \int \int (x - tv)^2 f(t, x, v) \, dv \, dx + g(t) \right] = \frac{g(t)}{t}, \quad t > 0. \]

Integration of this identity from 1 to \( t \geq 1 \) yields

\[ \int \int (x - tv)^2 f(t, x, v) \, dv \, dx + g(t) = C + \int_1^t \frac{g(s)}{s} \, ds \quad (6) \]

for some constant \( C > 0 \) which depends on \( f(1) \). We may drop the double integral and apply Gronwall’s lemma to the resulting inequality to obtain the estimate \( g(t) \leq Ct \) for \( t \geq 1 \), and this proves (2). Insertion of the estimate for \( g \) into (6) proves (3). Finally, for any \( R > 0 \) we have

\[ \rho(t, x) = \int_{|x-t=0| \leq R} f(t, x, v) \, dv + \int_{|x-t=0| \geq R} f(t, x, v) \, dv \]

\[ \leq \int_{|x-t=0| \leq R/\theta} f(t, x, v) \, dv + \int \frac{(x - tv)^2}{R^2} f(t, x, v) \, dv \]

\[ \leq \frac{4\pi}{3} \| f(t) \|_{\infty} \frac{R^3}{t^3} + \frac{1}{R^2} \int (x - tv)^2 f(t, x, v) \, dv. \]

Using the fact that \( f \) is constant along characteristics and thus \( \| f(t) \|_{\infty} = \| f(0) \|_{\infty} \) and optimizing the above estimate in \( R \) we obtain

\[ \rho(t, x) \leq Ct^{-6/5} \left( \int (x - tv)^2 f(t, x, v) \, dv \right)^{3/5} \]

for all \( t > 0 \) and \( x \in \mathbb{R}^3 \). Taking this to the power \( 5/3 \) and integrating with respect to \( x \) we complete the proof of (4).

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References


