Stability of spherically symmetric steady states in galactic dynamics against general perturbations

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Abstract

Certain steady states of the Vlasov-Poisson system can be characterized as minimizers of an energy-Casimir functional, and this fact implies a nonlinear stability property of such steady states. In previous investigations by Y. Guo and the author stability was obtained only with respect to spherically symmetric perturbations. In the present investigation we show how to remove this unphysical restriction.

1 Introduction

A classical problem in astrophysics is to determine which equilibrium configurations of self-gravitating ensembles of mass points, such as galaxies or globular clusters, are stable. Neglecting collisions and relativistic effects such ensembles are modeled by the Vlasov-Poisson system

\[
\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla f = \Delta f + U \nabla f = 0,
\]

(1.1)

\[
\frac{\partial U}{\partial t} = 4\pi \rho, \quad \lim_{|x| \to \infty} U(t,x) = 0,
\]

(1.2)
\[ \rho(t,x) = \int f(t,x,v) dv, \]  
(1.3)

where \( f = f(t,x,v) \geq 0 \) denotes the density of the stars in phase space, \( t \in \mathbb{R} \) denotes time, \( x, v \in \mathbb{R}^3 \) denote position and velocity respectively, \( \rho \) is the spatial mass density, and \( U \) the gravitational potential.

In the present paper we show how certain steady states of this system arise as minimizers of an energy-Casimir functional

\[ \mathcal{H}_C(f) := \frac{1}{2} \int \int |v|^2 f(x,v) dv dx - \frac{1}{8\pi} \int \int |\nabla U_f(x)|^2 dx + \int \int Q(f(x,v)) dv dx \]  
(1.4)

with a suitably chosen function \( Q \), and deduce nonlinear stability of these steady states against general perturbations. Similar variational techniques have been used for example for nonlinear Schrödinger equations, cf. [4]. For the Vlasov-Poisson system this approach was first used in a series of papers by Y. Guo and the author, cf. [7, 8, 9, 16]. However, these investigations were restricted to the case of spherical symmetry, and the aim of the present paper is to remove this restriction, since physically realistic perturbations, say by the gravitational pull of some distant galaxy, are not spherically symmetric.

To avoid the restriction to spherical symmetry we must minimize \( \mathcal{H}_C \) over not necessarily symmetric functions, and the main issue is to establish a compactness result along such general minimizing sequences. This is achieved using the scaling and splitting properties of the energy-Casimir functional in connection with a concentration-compactness principle due to P.-L. Lions.

By the variational equation the distribution function \( f_0 \) of a minimizer with induced potential \( U_0 \) is a function of the particle energy

\[ E := \frac{1}{2} |v|^2 + U_0(x), \]  
(1.5)

a conserved quantity along characteristics. For spherically symmetric steady states \( f_0 \) may also depend on

\[ L := |x|^2 |v|^2 - (x \cdot v)^2, \]  
(1.6)

the modulus of angular momentum squared which is also conserved along characteristics. To handle such steady states the function \( Q \) in (1.4) must also depend on \( L \), and [7, 8, 9] treated this more general case. However, in order to obtain stability \( \mathcal{H}_C \) must be conserved along solutions launched by
perturbations of \( f_0 \). For the sum of the first two terms in \( \mathcal{H}_C \), the kinetic and the potential energy of the system, this is true in general, but in the presence of \( L \) the third term in \( \mathcal{H}_C \), the so-called Casimir functional, is conserved only in the case of spherical symmetry. Thus, to allow for general, non-symmetric perturbations we exclude steady states which depend on \( L \).

To put the present investigation into perspective we compare the variational approach sketched above with other approaches. To this end we recall the well known class of polytropic steady states where

\[
f_0(x, v) = (E_0 - E)^k \sqrt{I}.
\]

Here \((\cdot)_+\) denotes the positive part, \( E_0 \in \mathbb{R} \) is a constant, and \( k > -1 \), \( l > -1 \), \( k + l + 1/2 > 0 \), \( k < 3l + 7/2 \); only for this range of exponents do these steady states have compact support and finite mass. The first nonlinear stability result for the Vlasov-Poisson system in the present stellar dynamics case is due to G. Wolansky [21]. It is restricted to spherically symmetric perturbations of the polytropes with exponents \( l > -1 \), \( 0 < k < l + 3/2 \) with \( k \neq -l - 1/2 \) and uses a variational approach for a reduced functional which is not defined on a set of phase space densities \( f \) but on a set of mass functions \( M(r) := \int_{|y| \leq r} \rho(y) \, dy \), \( r \geq 0 \) the radial coordinate. In particular, it does not yield a stability estimate for the phase space distribution \( f \). In [20] Y.-H. Wan proves stability by a careful investigation of the quadratic and and higher order parts in a Taylor expansion of \( \mathcal{H}_C \) about a steady state. He has to assume the existence of the steady state, requires a strong condition on \( f_0 \) which is satisfied by the polytropes only for \( k = 1 \) and \( l = 0 \), but he does not require spherical symmetry of the admissible perturbations. Finally, the approach in [7, 8, 9] gives the existence of the steady states (and actually provides new ones), covers the polytropes for \( l > -1 \) and \( 0 < k < l + 3/2 \), and, as we believe, has the simplest proof of the three approaches. With the present investigation we remove the only restriction this approach had so far when compared with [20], namely spherical symmetry of the admissible perturbations. We also mention [1] where stability for the limiting case \( k = 7/2 \) and \( l = 0 \) of polytropes with finite mass but infinite support is discussed.

The paper proceeds as follows: In the next section we establish some preliminary estimates which show that \( \mathcal{H}_C \) is bounded from below and the positive terms in \( \mathcal{H}_C \) are bounded along minimizing sequences. In Section 3 the existence of a minimizer of \( \mathcal{H}_C \) is established. Most of the technical
steps can be taken over from [8, 9], since in these papers spherical symmetry was only used to prevent mass from running off to spatial infinity along a minimizing sequence. To control this in the nonsymmetric case we use the concentration-compactness lemma due to P.-L. Lions. In Section 4 we show that minimizers are spherically symmetric steady states of the Vlasov-Poisson system with finite mass and compact support. The stability properties of the steady states are discussed in the last section. Here we point out one problem: If \( f_0 \) is a steady state then \( f_0(x + V t, v + V) \) for any given velocity \( V \in \mathbb{R}^3 \) is a solution of the Vlasov-Poisson system which for \( V \) small starts close to \( f_0 \), but travels away from \( f_0 \) at a linear rate in \( t \). This trivial “instability”, which cannot be present for spherically symmetric perturbations, has to be dealt with, and incidentally, both [20] and the present paper handle this by comparing \( f_0 \) with an appropriate shift in \( x \)-space of the time dependent perturbed solution \( f(t) \). In our case this shift arises from the application of the concentration-compactness lemma.

We conclude the introduction with some further references. Global classical solutions to the initial value problem for the Vlasov-Poisson system were first established in [13], cf. also [18]. Many references to discussions of the stability problem in the astrophysics literature can be found in the monograph [5]. A rigorous investigation of linearized stability is given in [2]. For the plasma physics case, where the sign in the Poisson equation (1.2) is reversed, the stability problem is easier and better understood, cf. [3, 10, 11, 15]. A quite general condition which guarantees finite mass and compact support of steady states, but not their stability, is established in [17].

2 Preliminaries

For a measurable function \( f = f(x,v) \) we define

\[
\rho_f(x) := \int f(x,v) dv, \quad x \in \mathbb{R}^3,
\]

and

\[
U_f := -\rho_f \ast \frac{1}{|\cdot|}.
\]

As to the existence of this convolution see Lemma 1 below. Next we define

\[
E_{\text{kin}}(f) := \frac{1}{2} \int \int |v|^2 f(x,v) dv dx,
\]
\[ E_{\text{pol}}(f) := -\frac{1}{8\pi} \int |\nabla U_f(x)|^2 dx = -\frac{1}{2} \iint \frac{\rho_f(x)\rho_f(y)}{|x-y|} dx dy, \]

\[ C(f) := \iint Q(f(x,v)) dv dx, \]

and

\[ \mathcal{H}_C(f) := C(f) + E_{\text{kin}}(f) + E_{\text{pol}}(f), \]

\[ \mathcal{P}(f) := C(f) + E_{\text{kin}}(f), \]

where \( Q \) is a given function satisfying certain assumptions specified below. Note that \( \mathcal{P} \) is the positive part of the energy-Casimir functional \( \mathcal{H}_C \). We will minimize \( \mathcal{H}_C \) over the set

\[ \mathcal{F}_M := \{ f \in L^1(\mathbb{R}^6) \mid f \geq 0, \iint f dv dx = M, \mathcal{P}(f) < \infty \}, \tag{2.1} \]

where \( M > 0 \) is prescribed. The function \( Q \) which determines the Casimir functional has to satisfy the following

**Assumptions on \( Q \):** \( Q \in C^1([0, \infty]) \cap C^2([0, \infty]), Q \geq 0, \) and there exist constants \( C_1, C_2 > 0, F_0 > 0, \) and \( 0 < k_1, k_2, k_3 < 3/2 \) such that:

1. \( (Q1) \ Q(f) \geq C_1 f^{1+1/k_1}, \ f \geq F_0. \)
2. \( (Q2) \ Q(f) \leq C_2 f^{1+1/k_2}, \ 0 \leq f \leq F_0. \)
3. \( (Q3) \ Q(\lambda f) \geq \lambda^{1+1/k_3} Q(f), \ f \geq 0, \ 0 \leq \lambda \leq 1. \)
4. \( (Q4) \ Q'(f) > 0, \ f > 0, \) and \( Q'(0) = 0. \)

On their support the steady states obtained later will be of the form

\[ f_0(x,v) = (Q')^{-1}(E_0 - E) \]

with some \( E_0 < 0 \) and \( E \) as defined in \( (1.5) \); under the assumptions above \( Q' \) is strictly increasing with range \([0, \infty] \). A typical example of a function \( Q \) satisfying the assumptions is

\[ Q(f) = f^{1+1/k}, \ f \geq 0, \tag{2.2} \]

with \( 0 < k < 3/2 \) which leads to a steady state of polytropic form \( (1.7) \).

We collect some estimates for \( \rho_f \) and \( U_f \) induced by an element \( f \in \mathcal{F}_M \). As in the rest of the paper constants denoted by \( C \) are positive, may depend on \( M \) and \( Q \), and may change their value from line to line.

5
Lemma 1 Let \( n_1 := k_1 + 3/2 \) so that \( 1 + 1/n_1 > 4/3 > 6/5 \). Then for any \( f \in \mathcal{F}_M \) the following holds:

(a) \( f \in L^{1+1/k_1}(\mathbb{R}^6) \) with

\[
\iint f^{1+1/k_1} dv \, dx \leq C (1 + \mathcal{P}(f)).
\]

(b) \( \rho_f \in L^{1+1/m_1}(\mathbb{R}^3) \) with

\[
\int \rho_f^{1+1/m_1} dx \leq C (1 + \mathcal{P}(f)).
\]

(c) \( U_f \in L^6(\mathbb{R}^3) \) with \( \nabla U_f \in L^2(\mathbb{R}^3) \), and

\[
\int |\nabla U_f|^2 dx \leq C \|\rho_f\|_{6/5}^2 \leq C \|\rho_f\|_{1+1/m_1}^{n_1}.\]

The two representations of \( E_{pot}(f) \) stated above are indeed equal.

Proof. Part (a) follows immediately from (Q1). To obtain (b) we split the \( v \)-integral defining \( \rho_f \) into small and large velocities and optimize with respect to the split, cf. [8, Lemma 1] or [9, Lemma 1].

For the convenience of the reader we recall here the extended Young’s inequality, cf. [14, p. 32]: Let \( 1 < p, q, r < \infty \) with \( 1/p + 1/q = 1 + 1/r \). Then for all \( g \in L^p(\mathbb{R}^n) \), \( h \in L_w^q(\mathbb{R}^n) \) we have \( g \ast h \in L^r(\mathbb{R}^n) \) with

\[
\|g \ast h\|_r \leq \|g\|_p \|h\|_{q, w}.
\]  

(2.3)

Here \( h \in L_w^q(\mathbb{R}^n) \) iff \( h \) is measurable and

\[
\|h\|_{q, w} := \sup_{s > 0} \left[ s^q \text{vol} \{ x \in \mathbb{R}^n \mid |h(x)| > s \} \right]^{1/q} < \infty.
\]

The estimates for \( U_f \) and \( \nabla U_f \) in (c) follow with \( n = 3, \ p = 6/5, \) and \( h = |\cdot|^{-1}, \ q = 3 \) or \( h = |\cdot|^{-2}, \ q = 3/2 \). Integration by parts—after regularizing \( \rho_f \) if necessary—shows that the two formulas for \( E_{pot}(f) \) are equal. \( \Box \)

An immediate corollary of the lemma above one can show that on \( \mathcal{F}_M \) the functional \( \mathcal{H}_C \) is bounded from below in such a way that \( \mathcal{P} \)—and thus certain norms of \( f \) and \( \rho_f \)—remain bounded along minimizing sequences—note that \( n_1 < 3 \):
Lemma 2  For every $M > 0$ there exists a constant $C > 0$ such that
\[ \mathcal{H}_C(f) \geq \mathcal{P}(f) - C(1 + \mathcal{P}(f))^{n_1/3}, \ f \in \mathcal{F}_M, \]
in particular,
\[ h_M := \inf_{\mathcal{F}_M} \mathcal{H}_C > -\infty. \]

3  Existence of minimizers

The behaviour of $\mathcal{H}_C$ and $M$ under scaling transformations can be used to show that $h_M$ is negative and to relate the $h_M$’s for different values of $M$:

Lemma 3  (a) Let $M > 0$. Then $-\infty < h_M < 0$.

(b) There exists $\alpha > 0$ such that for all $0 < M_1 \leq M_2$,
\[ h_{M_1} \geq \left( \frac{M_1}{M_2} \right)^{1+\alpha} h_{M_2}. \]

For the proof we refer to [8, Lemma 4] or [9, Lemma 4]; spherical symmetry was not used in those proofs, cf. also [16, Lemma 4] where we explicitly kept track of where symmetry was used. A simple consequence of part (b) of the lemma above is that
\[ h_M < h_{M-m} + h_m, \ 0 < m < M, \]
which is condition (S.2) in [12, Theorem II.1], but we prefer to work with (b).

For easier reference we state the concentration-compactness lemma which replaces the splitting estimates used in [8, 9]. This is Lemma 1.1 in [12], cf. also [19, 4.3]. The fact that we have functions of two variables $x$ and $v$ but consider the various balls
\[ B_R := \{ x \in \mathbb{R}^3 \mid |x| \leq R \} \]
only in $x$-space requires no changes in the proof.

Lemma 4  Let $(f_n) \subset L^1(\mathbb{R}^6)$ with $f_n \geq 0$ and $\int f_n = M, \ n \in \mathbb{N}$. Then there exists a subsequence $(f_{n_k})$ such that one of the following assertions holds:
(i) \( \exists (a_k) \subset \mathbb{R}^3 \ \forall \epsilon > 0 \ \exists R > 0, \ k_0 \in \mathbb{N} : \)
\[ \int_{a_k + B_R} f_{n_k} \, dv \, dx \leq M - \epsilon, \ k \geq k_0. \]

(ii) \( \forall R > 0 : \)
\[ \lim_{k \to \infty} \sup_{a \in \mathbb{R}^3} \int_{a + B_R} f_{n_k} \, dv \, dx = 0. \]

(iii) \( \exists m \in [0, M] [ \ \forall \epsilon > 0 \ \exists k_0 \in \mathbb{N}, \ (f^1_k), \ (f^2_k) \subset L^1(\mathbb{R}^6) : \)
\[ \| f_{n_k} - (f^1_k + f^2_k) \|_1 \leq \epsilon, \ \left| \int f^1_k - m \right| \leq \epsilon, \ \left| \int f^2_k - (M - m) \right| \leq \epsilon, \ k \geq k_0. \]

and
\[ \text{dist} \left( \text{supp} f^1_k, \text{supp} f^2_k \right) \to \infty, \ k \to \infty, \]

where \( g_k := f_{n_k} - f^1_k - f^2_k. \)

In the case of a minimizing sequence for \( \mathcal{H}_C \), Lemma 3 can be used to exclude possibilities (ii) and (iii) in the previous lemma:

**Lemma 5** Let \( (f_n) \subset \mathcal{F}_M \) be a minimizing sequence of \( \mathcal{H}_C \). Then in Lemma 4 only (i) holds.

**Proof.** For \( R > 1 \) define
\[ K_R(x) := \begin{cases} \frac{1}{|x|}, & 1/R \leq |x| \leq R, \\ R, & |x| < 1/R, \\ 0, & |x| > R, \end{cases} \]

and
\[ F_R(x) := \frac{1}{|x|} 1_{\{|x| > R\}}(x), \ G_R(x) := \left( \frac{1}{|x|} - R \right) 1_{\{|x| < 1/R\}}(x) \]

so that
\[ \frac{1}{|x|} = K_R(x) + F_R(x) + G_R(x), \ x \in \mathbb{R}^3. \] (3.1)
Here $1_A$ denotes the indicator function of the set $A$. Assume (ii) holds and split

$$
\frac{1}{4\pi} \int |\nabla U_n|^2 dx = \int \int \frac{\rho_n(x)\rho_n(y)}{|x-y|} dy \, dx = I_1 + I_2 + I_3
$$

according to (3.1). Since $(\rho_n)$ is bounded in $L^{4/3}(\mathbb{R}^3)$ and $\|\rho_n\|_1 = M$, $n \in \mathbb{N}$, we find

$$
|I_1| \leq R \int \int_{|x-y| < R} \rho_n(x) \rho_n(y) \, dx \, dy \leq RM \sup_{y \in \mathbb{R}^3} \int_{y + B_R} \rho_n(x) \, dx,
$$

$$
|I_2| \leq \frac{1}{R} \int \int \rho_n(x) \rho_n(y) \, dx \, dy = M^2 R^{-1},
$$

$$
|I_3| \leq \|\rho_n\|_{4/3} \|\rho_n \ast G_R\|_4 \leq C \|\rho_n\|_{4/3}^2 \|G_R\|_2 \leq CR^{-1/2};
$$

for the last term we used Hölder’s and Young’s inequality, cf. (2.3). Since this holds for any $R > 1$, we conclude by (ii) that $E_{\text{pot}}(f_{n_k}) \to 0$ along the subsequence obtained in Lemma 4. Hence $h_M \geq 0$, a contradiction to Lemma 3 (a).

Assume that (iii) holds. We denote the subsequence obtained in Lemma 4 by $(f_n)$. Let $m \in ]0, M[$ be according to (iii) and $\epsilon > 0$ arbitrary. With $m_n := \int \rho_n^1$, $M_n := \int \rho_n^2$, obvious definitions for $\rho_n^1$ and $\sigma_n := \int g_n \, dv$ we have

$$
|m_n - m| \leq \epsilon, \quad |M_n - (M - m)| \leq \epsilon
$$

and

$$
\mathcal{P}(f_n) = \mathcal{P}(f_n^1 + f_n^2 + g_n) = \mathcal{P}(f_n^1) + \mathcal{P}(f_n^2) + \mathcal{P}(g_n) \geq \mathcal{P}(f_n^1) + \mathcal{P}(f_n^2).
$$

Moreover

$$
E_{\text{pot}}(\rho_n) = E_{\text{pot}}(\rho_n^1) + E_{\text{pot}}(\rho_n^2) - I_1 - I_2 + I_3
$$

where

$$
I_1 := \int \int \frac{\rho_n^1(x)\rho_n^2(y)}{|x-y|} \, dx \, dy,
$$

$$
I_2 := \int \int \frac{\rho_n(x)\sigma_n(y)}{|x-y|} \, dx \, dy,
$$

$$
I_3 := \frac{1}{2} \int \int \frac{\sigma_n(x)\sigma_n(y)}{|x-y|} \, dx \, dy.
$$
To estimate $I_1$ observe that for $n$ sufficiently large, dist $(\text{supp} \rho_n^1, \text{supp} \rho_n^2) > 1/\epsilon$ so that

$$|I_1| \leq M^2 \epsilon.$$

To estimate $I_2$ use the extended Young’s inequality (2.3) and interpolation to find

$$|I_2| \leq C \|\rho_n\|_{6/5} \|\sigma_n\|_{6/5} \leq C \|\sigma_n\|_1^{1/3} \leq C \epsilon^{1/3}.$$

As to $I_3$ it suffices to observe that this term is nonnegative. Thus for any $\epsilon < 1$ and all sufficiently large $n$ we find, using Lemma 3 (b),

$$h_M \geq \mathcal{P}(f_n) + E_{\text{pot}}(f_n) - \epsilon \geq \mathcal{P}(f_n^1) + \mathcal{P}(f_n^2) + E_{\text{pot}}(f_n^1) + E_{\text{pot}}(f_n^2) - C \epsilon^{1/3} \geq h_m + h_{M_n} - C \epsilon^{1/3} \geq \left[ \left( \frac{m_1}{M} \right)^{1+\alpha} + \left( \frac{M_n}{M} \right)^{1+\alpha} \right] h_1 - C \epsilon^{1/3};$$

clearly $0 < m_n, M_n < M$ for $\epsilon > 0$ sufficiently small. To continue we define

$$C_\alpha := \inf_{x \in [0,1]} \frac{(1-x)^{1+\alpha} + x^{1+\alpha} - 1}{(1-x)x} > 0.$$

Since $h_M < 0$ it follows that

$$1 \leq \left( \frac{m_1}{M} \right)^{1+\alpha} + \left( \frac{M_n}{M} \right)^{1+\alpha} + C \epsilon^{1/3} \leq \left( \frac{m_1}{M} \right)^{1+\alpha} + \left( \frac{M-m_1}{M} \right)^{1+\alpha} + C \epsilon^{1/3} \leq \left( \frac{m_1}{M} \right)^{1+\alpha} - \left( \frac{m_1}{M} \right)^{1+\alpha} + \left( \frac{M-n}{M} \right)^{1+\alpha} - \left( \frac{M-n}{M} \right)^{1+\alpha} \leq 1 - C_\alpha \left( \frac{m_1}{M} \right)^{1+\alpha} + C \epsilon^{1/3} + 2 \left( \frac{1+\alpha}{M} \right) \epsilon, \, \epsilon \in [0,1],$$

and this is a contradiction. Thus only assertion (i) can hold. \hfill \Box

**Theorem 1** Let $M > 0$. Let $(f_n) \subset \mathcal{F}_M$ be a minimizing sequence of $\mathcal{H}_C$. Then there is a minimizer $f_0 \in \mathcal{F}_M$, a subsequence $(f_{n_k})$, and a sequence $(a_k) \subset \mathbb{R}^3$ such that

$$\mathcal{H}_C(f_0) = \inf_{\mathcal{F}_M} \mathcal{H}_C =: h_M.$$
and $f_{n_k} \to f_0$ weakly in $L^{1+1/k_1}(\mathbb{R}^6)$. For the induced potentials we have
$\nabla \mathcal{U}_{n_k} \to \nabla \mathcal{U}_0$ strongly in $L^2(\mathbb{R}^3)$. Here $f^a(x,v) := f(x+a,v)$.

**Proof.** Let $(f_n)$ be a minimizing sequence. Use Lemma 5 to choose a subsequence, denoted by $(f_n)$ again and a sequence $(a_n) \subset \mathbb{R}^3$ such that (i) in Lemma 4 holds. Let $\bar{f}_n(x,v) := f_n(x+a_n,v)$. This is again a minimizing sequence, because $\mathcal{H}_C$ is translation invariant. By Lemma 2, $(\mathcal{P}(\bar{f}_n))$ is bounded and thus $(\bar{f}_n)$ is bounded in $L^{1+1/k_1}(\mathbb{R}^6)$. Thus there exists a weakly convergent subsequence, denoted by $(\bar{f}_n)$ again:

$$\bar{f}_n \rightharpoonup f_0 \text{ weakly in } L^{1+1/k_1}(\mathbb{R}^6).$$

Clearly, $f_0 \geq 0$ a.e. By Lemma 1 $(\bar{p}_n) = (\rho_{\bar{f}_n})$ is bounded in $L^{1+1/k_1}(\mathbb{R}^3)$. After extracting a further subsequence

$$\bar{p}_n \rightharpoonup \rho_0 := \rho_{f_0} \text{ weakly in } L^{1+1/k_1}(\mathbb{R}^3).$$

Also by weak convergence $E_{\text{kin}}(f_0) \leq \liminf_{n \to \infty} E_{\text{kin}}(\bar{f}_n)$. By (Q4) the functional $\mathcal{C}$ is convex. Thus by Mazur’s Lemma and Fatou’s Lemma

$$\mathcal{C}(f_0) \leq \limsup_{n \to \infty} \mathcal{C}(\bar{f}_n),$$

and

$$\mathcal{P}(f_0) \leq \limsup_{n \to \infty} \mathcal{P}(\bar{f}_n).$$

We show that $f_0 \in \mathcal{F}_M$. Let $\epsilon > 0$. By (i) in Lemma 4 and the boundedness of $E_{\text{kin}}(f_n)$ there exists $R > 0$ such that

$$M \geq \int_{B_R} f_0 dv dx \geq M - \epsilon$$

which implies that $\int f f_0 = M$ and $f_0 \in \mathcal{F}_M$. It remains to deal with the potential energy:

$$\frac{1}{8\pi} \int \left| \nabla \mathcal{U}_{\bar{f}_n} - \nabla \mathcal{U}_0 \right|^2 dx = \frac{1}{2} \int \int \frac{(\bar{p}_n(x) - \rho_0(x))(\bar{p}_n(y) - \rho_0(y))}{|x-y|} dx dy = I_1 + I_2 + I_3$$

where the latter integral is split according to (3.1).

11
Estimate of \( I_1 \): Define
\[
U_n^R := -\bar{\rho}_n * K_R, \quad U_0^R := -\rho_0 * K_R, \quad R > 0.
\]
Since \( K_R \in L^1(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3) \) weak convergence of \( \bar{\rho}_n \) implies that for any \( R > 0 \),
\[
U_n^R \to U_0^R, \quad n \to \infty, \quad \text{pointwise on } \mathbb{R}^3.
\]
Since \( \int U_n^R = \int U_0^R \) we find \( U_n^R \to U_0^R \) in \( L^1(\mathbb{R}^3) \) as \( n \to \infty \) for any \( R > 0 \). Now Hölder’s inequality, an interpolation argument, and Young’s inequality together with the boundedness of the \( \rho \)'s in \( L^{4/3}(\mathbb{R}^3) \) imply that
\[
|I_1| \leq \|\bar{\rho}_n - \rho_0\|_{4/3} \|U_n^R - U_0^R\|_4 \leq C \|U_n^R - U_0^R\|_{12}^{9/11} \|U_n^R - U_0^R\|_1^{2/11} \\
\leq C \|\bar{\rho}_n - \rho_0\|_{4/3} \|U_n^R - U_0^R\|_1^{2/11} \to 0, \quad n \to \infty, \quad R > 0.
\]
Obviously
\[
|I_2| \leq \frac{4M^2}{R},
\]
and again by Hölder’s and Young’s inequality
\[
|I_3| \leq \|\bar{\rho}_n - \rho_0\|_{4/3} \|\bar{\rho}_n - \rho_0\| \leq C \|G_R\| \leq CR^{-1/2}.
\]
Thus \( \nabla \bar{U}_n \to \nabla U_0 \) in \( L^2(\mathbb{R}^3) \) for \( n \to \infty \), and the proof is complete. \( \square \)

4 Properties of minimizers

Theorem 2 Let \( f_0 \in \mathcal{F}_M \) be a minimizer of \( \mathcal{H}_C \). Then
\[
f_0(x, v) = \begin{cases} 
(Q')^{-1}(E_0 - E), & E_0 - E > 0, \\
0, & E_0 - E \leq 0
\end{cases}
\]
where
\[
E := \frac{1}{2}|v|^2 + U_0(x),
\]
\[
E_0 := \frac{1}{M} \int \int (Q'(f_0) + E) f_0 \, dv \, dx,
\]
and \( U_0 \) is the potential induced by \( f_0 \). In particular, \( f_0 \) is a steady state of the Vlasov-Poisson system.
For the proof we refer to [8, Thm. 2] where a somewhat stronger condition (Q4) was used or [9, Thm. 2] where (Q4) is as in the present paper; if anything, the fact that we do not require spherical symmetry of the functions in $\mathcal{F}_M$ makes the proof of this theorem easier.

There now arise a couple of questions which are all interrelated: Firstly, in which sense does $f_0$ satisfy the stationary Vlasov-Poisson system? Up to now, the Poisson equation holds in the sense of distributions, and the Vlasov equation in the sense that $f_0$ is constant along characteristics, but $\nabla U_0$ is not sufficiently regular to define classical characteristics to begin with. Secondly, we know that if we minimize $\mathcal{H}_\mathcal{C}$ over the space of spherically symmetric functions in $\mathcal{F}_M$ we obtain a spherically symmetric minimizer with compact support. Are the minimizers that we obtain in the present, more general context still spherically symmetric and compactly supported? Are they unique? These questions are considered next; $C_b^k$ and $C^k_b$ denote the space of $C^k$ functions with compact support or with bounded derivatives up to order $k$, respectively:

**Theorem 3** Let $f_0 \in \mathcal{F}_M$ be a minimizer of $\mathcal{H}_\mathcal{C}$ so that by Theorem 2 $f_0(x,v) = \phi(E)$ with $\phi$ determined by $Q$.

(a) Assume that

$$\phi(E) \leq C_1'(E_0 - E)^{k_1}, \quad E \to -\infty$$

and

$$\phi(E) \geq C_2'(E_0 - E)^{k_2}, \quad E \to E_0^-$$

for positive constants $C_1'$, $C_2'$ (as is illustrated by the polytropes these assumptions are compatible with the general assumptions on $Q$). Then $E_0 < 0$, $\rho_0 \in C_1^1(\mathbb{R}^3)$, $U_0 \in C_2^3(\mathbb{R}^3)$ with $\lim_{|x| \to \infty} U_0(x) = 0$, and the steady state is spherically symmetric with respect to some point in $\mathbb{R}^3$.

(b) If in particular $Q(f) = f^{1+1/k}$, $f \geq 0$, with $0 < k < 3/2$ then up to a shift in $x$-space the minimizer is unique in $\mathcal{F}_M$.

**Proof.** To prove part (a) the basic idea is to use Sobolev embedding to obtain the desired regularity, establish the appropriate behaviour of $U_0$ at infinity, and then apply a result by Gidas, Ni and Nirenberg to conclude the spherical symmetry, cf. [6, Thm. 4]. First we show that

$$-U_0(x) \geq \frac{M}{3|x|}, \quad |x| \to \infty. \quad (4.1)$$

13
To see this, choose $R > 0$ such that
\[
\int_{|y| \leq R} \rho_0(y) \, dy > \frac{M}{2}.
\]
Since
\[
\left| \frac{1}{|x-y|} - \frac{1}{|x|} \right| \leq \frac{R}{(|x|-R)^2}, \quad |x| \geq 2R, \ |y| \leq R
\]
we obtain (4.1) by restricting the convolution integral defining $U_0$ to the ball \( \{ |y| \leq R \} \) and expanding the kernel as indicated.

Next we claim that
\[
\rho_0 \in L^4(\mathbb{R}^3). \tag{4.2}
\]
To see this, we observe that $f_0$ depends only on the particle energy $E$ via the function $\phi$, and thus
\[
\rho_0(x) = h_\phi(U_0(x)), \ x \in \mathbb{R}^3 \tag{4.3}
\]
where
\[
h_\phi(u) := 4\pi \sqrt{2} \int_u^\infty \phi(E) \sqrt{E-u} \, du, \ u \in \mathbb{R}; \tag{4.4}
\]
note that $h_\phi(u) = 0$ for $u \geq E_0$. The general assumptions on $Q$ and the additional assumption in the theorem imply that
\[
h_\phi(u) \leq C \left( 1 + (E_0 - u)^{k_1+3/2} \right), \ u \leq E_0.
\]
If we use this estimate on the set where $\rho_0$ is large—this set has finite measure—and the integrability of $\rho_0$ on the complement we find that
\[
\int \rho_0(x)^p \, dx \leq C \int (-U_0(x))^{(k_1+3/2)p} \, dx,
\]
and since by Lemma 1 (c) $U_0 \in L^{12}(\mathbb{R}^3)$ this is finite for $p = 12/(k_1+3/2) > 4$.

The next step is to show that
\[
U_0 \in L^\infty(\mathbb{R}^3), \ U_0(x) \to 0, \ |x| \to \infty. \tag{4.5}
\]
To see this we split the potential in the following way:
\[
-U_0(x) = \int_{|x-y|<1/R} \frac{\rho_0(y)}{|x-y|} \, dy + \int_{1/R \leq |x-y|<R} \cdots + \int_{|x-y| \geq R} \cdots
\]
\[
\leq C \|\rho_0\|_4 \left( \int_0^{1/R} r^{2-4/3} \, dr \right)^{3/4} + R \int_{|y| \geq |x|-R} \rho_0(y) \, dy + \frac{M}{R}, \ |x| \geq R,
\]
14
and since \( R > 1 \) is arbitrary and \( \rho_0 \in L^1(\mathbb{R}^3) \cap L^4(\mathbb{R}^3) \) assertion (4.5) follows.

We are now in the position to show that

\[
E_0 < 0 \text{ and supp } \rho_0 \text{ is compact.}
\] (4.6)

By (4.3) and (4.5) the second assertion follows from the first. Assuming that \( E_0 > 0 \) immediately implies that \( \rho_0(x) \geq h_\phi(E_0/2) > 0 \) for all sufficiently large \( |x| \) which contradicts the integrability of \( \rho_0 \). Assume that \( E_0 = 0 \). Then the estimate for \( \phi \) from below implies

\[
h_\phi(u) \geq C(-u)^{k_2+3/2}, \ u \to 0 -.
\]

But then (4.1) implies that

\[
\rho_0(x) \geq C|x|^{-k_2-3/2}
\]

for all sufficiently large \( |x| \), and since \( k_2 + 3/2 < 3 \) this again contradicts the integrability of \( \rho_0 \). Thus only the alternative \( E_0 < 0 \) remains, and (4.6) is established.

Next we establish the desired regularity of the steady state. Since \( U_0 \in L^\infty(\mathbb{R}^3) \) this is also true for \( \rho_0 \), cf. (4.3). This in turn implies that the first order derivatives of \( U_0 \) are bounded, i.e., \( U_0 \in W^{1,\infty}(\mathbb{R}^3) \). By Sobolev embedding \( U_0 \in C_b(\mathbb{R}^3) \), thus \( \rho_0 \in C_c(\mathbb{R}^3) \). This in turn implies that \( U_0 \in C^1_b(\mathbb{R}^3) \), thus \( \rho_0 \in C^1_c(\mathbb{R}^3) \), and thus \( U_0 \in C^2(\mathbb{R}^3) \). Observe that the function \( h_\phi \) defined in (4.4) is continuously differentiable.

If we define \( V := -U_0 > 0 \) and expand \( 1/|x-y| \) in powers of \( y \) to third order for \( y \in \text{supp } \rho_0 \) and \( |x| \) large we find that the assumptions in [6, Thm. 4] hold. Thus \( U_0 \) is spherically symmetric about some point in \( \mathbb{R}^3 \), and the proof of part (a) is complete.

As to part (b) we first observe that up to some shift \( U_0 \) as a function of the radial variable \( r := |x| \) solves the equation

\[
\frac{1}{r^2} (r^2 U_0')' = c_k (E_0 - U_0)^{k_2+3/2}, \ r > 0,
\] (4.7)

with some appropriately defined constant \( c_k \). Here \( ' \) denotes the derivative with respect to \( r \). The function \( E_0 - U_0 \) is a solution of the singular ordinary differential equation

\[
\frac{1}{r^2} (r^2 z')' = -c_k z^{k_2+3/2}, \ r > 0.
\] (4.8)
Now observe that solutions $z \in C([0, \infty) \cap C^2([0, \infty))$ of (4.8) with $z'$ bounded near $r = 0$ are uniquely determined by $z(0)$. This is due to the fact that for such a solution the equation implies that $z'(0)$ exists and is zero; clearly, $U'_0(0) = 0$. Moreover, if $z$ is such a solution then so is
\[ z_\alpha(r) := \alpha z(\alpha^\gamma r), \quad r \geq 0 \]
for any $\alpha > 0$ where $\gamma := (k + 1/2)/2$, and $z_\alpha(0) = \alpha z(0)$. Now assume there exists another minimizer in $\mathcal{F}_M$, i.e., up to a shift another solution $U_1$ of (4.7) with cut-off energy $E_1 < 0$. Uniqueness for (4.8) yields some $\alpha > 0$ such that
\[ E_1 - U_1(r) = \alpha E_0 - \alpha U_0(\alpha^\gamma r), \quad r \geq 0. \]
However, both steady states have the same total mass $M$, so that
\[
M = c_k \int_0^\infty r^2 (E_1 - U_1(r))^{k+3/2} dr = \alpha^{k+3/2-3\gamma} c_k \int_0^\infty r^2 (E_0 - U_0(r))^{k+3/2} dr = \alpha^{k+3/2-3\gamma} M.
\]
Since the exponent of $\alpha$ is not zero, this implies that $\alpha = 1$, and considering limits at spatial infinity we conclude that $E_0 = E_1$ and $U_0 = U_1$. \hfill \Box

5 Dynamical stability

Before we start to investigate the dynamical stability of $f_0$ we recall that for $f(0) \in C^1_c(\mathbb{R}^6)$ there exists a unique global classical solution to the corresponding initial value problem, cf. [13, 18]. $\mathcal{H}_C$ is constant along this solution, and $f(t) \in \mathcal{F}_M$, $t \geq 0$, provided $f(0) \in \mathcal{F}_M$.

In order to derive a stability estimate from the conservation of $\mathcal{H}_C$ along solutions we note that
\[
\mathcal{H}_C(f) - \mathcal{H}_C(f_0) = d(f, f_0) - \frac{1}{8\pi} \|\nabla U_f - \nabla U_0\|^2, \quad f \in \mathcal{F}_M \tag{5.1}
\]
where
\[
d(f, f_0) := \int\int [Q(f) - Q(f_0) + (E - E_0)(f - f_0)] dv dx.
\]
Moreover, a simple Taylor expansion shows that
\[
d(f, f_0) \geq 0, \quad f \in \mathcal{F}_M
\]
always, and under further restrictions on $Q$, say for $Q(f) = f^{1+1/k}$ with $1 \leq k < 3/2$, we even have

$$d(f, f_0) \geq C\|f - f_0\|_2^2, \quad f \in \mathcal{F}_M.$$  

**Theorem 4** Assume that the minimizer $f_0$ is unique in $\mathcal{F}_M$. Then for every $\epsilon > 0$ there is a $\delta > 0$ such that for any solution $t \mapsto f(t)$ of the Vlasov-Poisson system with $f(0) \in C^1_c(\mathbb{R}^6) \cap \mathcal{F}_M$,

$$d(f(0), f_0) + \frac{1}{8\pi} \|\nabla U_f(0) - \nabla U_0\|_2^2 < \delta$$

implies that for every $t \geq 0$ there exists $a \in \mathbb{R}^3$ such that

$$d(f^n(t), f_0) + \frac{1}{8\pi} \|\nabla U_{f^n(t)} - \nabla U_0\|_2^2 < \epsilon.$$  

**Proof.** Assume the assertion of the theorem were false. Then there exist $\epsilon > 0$, $t_n > 0$, and $f_n(0) \in C^1_c(\mathbb{R}^6) \cap \mathcal{F}_M$ such that

$$d(f_n(0), f_0) + \frac{1}{8\pi} \|\nabla U_{f_n(0)} - \nabla U_0\|_2^2 \to 0, \quad n \to \infty,$$

but

$$\inf_{a \in \mathbb{R}^3} \left( d(f_n(t_n), f_0) + \frac{1}{8\pi} \|\nabla U_{f_n(t_n)} - \nabla U_0\|_2^2 \right) \geq \epsilon, \quad n \in \mathbb{N}. \quad (5.2)$$

Since $\mathcal{H}_C$ is conserved, (5.1) implies that

$$\lim_{n \to \infty} \mathcal{H}_C(f_n(t_n)) = \lim_{n \to \infty} \mathcal{H}_C(f_n(0)) = h_M,$$

i. e., $(f_n(t_n)) \subseteq \mathcal{F}_M$ is a minimizing sequence of $\mathcal{H}_C$. By Theorem 1, we deduce that—up to a subsequence—$\|\nabla U_{f_n(t_n)} - \nabla U_0\|_2^2 \to 0$. Since $(f_n(t_n))$ is a minimizing sequence as well, (5.1) implies that $d(f_n(t_n), f_0) \to 0$, a contradiction to (5.2). \hfill \Box

**Final remarks**

(a) We do in general have no control over the shift vectors $a$. One might think that taking only initial data with

$$\int \int v f(x,v) dv dx = \int \int x f(x,v) dv dx = 0 \quad (5.3)$$

17
might avoid the necessity of the shifts, since this condition propagates and eliminates the trivial instability due to perturbations of the form $f_0(x + iV, v + V)$ with $V \in \mathbb{R}^3$ fixed. However, as pointed out in [20], it is conceivable that an appropriate, small perturbation causes a small fraction of the total mass distribution to move off in one direction and the bulk of the distribution in the other direction in such a way that (5.3) holds, but one still has to shift the reference frame with the bulk of the distribution to save the stability estimate.

(b) If we restrict the set $F_M$ to spherically symmetric functions then clearly all shift vectors $a = 0$, and we recover the results in [8, 9] for the $L$-independent case.

(c) The question whether steady states which depend on angular momentum $L$ are stable against nonsymmetric perturbations remains open, since it is then no longer true that $\mathcal{H}_C$ is conserved along nonsymmetric solutions.

(d) Another open problem is the uniqueness of the minimizers if $Q$ is not of the polytropic form (2.2). We have found no substitute for the scaling argument used to analyze solutions of the equation (4.8) in the general case. However, should the minimizer not be unique (not even locally) then one still obtains a stability result in the sense that the whole set of minimizers is stable, cf. [8].

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References


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20